On the Smooth Points of T-stable Varieties in G/B and the Peterson Map

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Abstract

Let G be a semi-simple algebraic group over $\mathbb{C}$, B a Borel subgroup of G and T a maximal torus in B. A beautiful unpublished result of Dale Peterson says that if G is simply laced, then every rationally smooth point of a Schubert variety X in G/B is nonsingular in X. The purpose of this paper is to generalize this result to arbitrary T-stable subvarieties of G/B, the only restriction being that G contains no G_2 factors. A key idea in Peterson's proof is to deform the tangent space $T_y(X)$ to X at a nonsingular T-fixed point y along the orbit of y under a root subgroup in B, which is open in a T-invariant curve C (we say a T-curve) in X. In more generality, if a T-variety X is nonsingular along the open T-orbit in a T-curve C and $x \in C^T$, we may consider the limit $\tau_C(X,x)$ of the tangent spaces $T_z(X)$ as z approaches x along C. We call $\tau_C(X,x)$ the *Peterson translate* of X at x along C. Peterson showed that a Schubert variety X in G/B, where G is semi-simple, is nonsingular at $x \in X^T$ as long as all $\tau_C(X,x)$ coincide for all such *good* T-curves.. Our first result generalizes this theorem to any irreducible T-variety X, provided the fixed point x is attractive under much weaker hypotheses. We then prove that if X is a T-variety in G/B where G contains no G_2 factors, then every Peterson translate $\tau_C(X,x)$ is contained in the linear span $\Theta_x(X)$ of the reduced tangent cone to X at x. (This fails when $G = G_2$.) Combining these two results leads to our characterization of the smooth T-fixed points of such X. In particular, we show that if G is simply laced, then X is nonsingular at a T-fixed point x if and only if it is rationally smooth at x and x lies on at least two good T-curves Peterson's ADE result is an immediate consequence of this. In addition, we obtain a uniform description of the nonsingular T-fixed points of a Schubert variety in G/B modulo the G_2 restriction. In particular, a Schubert variety X in such a G/B is nonsingular if and only if all the reduced tangent cones of X are linear. Finally, we also obtain versions of the above results for all algebraic homogeneous spaces G/P modulo the G_2 restrictions.

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1. Introduction

Let G be a semi-simple algebraic group over $k = \mathbb{C}$. Fix a Borel subgroup B of G and a maximal torus $T \subset B$. The purpose of this paper is to investigate the singular locus of a T-stable subvariety X of the flag variety G/B. More precisely, we would like to describe the set of nonsingular T-fixed points of X. This problem originates with the question of determining the connection between the singular loci of a Schubert variety $X \subset G/B$ (i.e. the closure of a B-orbit) in the sense of rational smoothness (cf. [8, 9]) and the sense of algebraic geometry. It was shown, for example, in [6] that when G is of type A, i.e. G/B is the variety of complete flags in k^n, then the two singular loci are the same, in particular every rationally smooth point of X is nonsingular. More recently, Dale Peterson (unpublished) extended this to Schubert varieties in G/B in the full ADE setting (see below). His method is to study how tangent spaces of a Schubert variety X behave when deformed along T-invariant curves in X containing a nonsingular point of X.

Before describing our results, we need to fix some notation. Recall that the T-fixed point set G/B^T is in a one to one correspondence with the Weyl group W of (G,T) via $w \mapsto wB$. Hence we may simply denote $wB \in G/B^T$ by w. The Schubert variety $X(w)$ associated to $w \in W$ is by definition the Zariski closure in G/B of the B-orbit Bw. Recall that B defines a Coxeter system for W. Let $\leq$ denote the associated partial order on W, the so called Bruhat-Chevalley order. This Coxeter system has two fundamental properties. Firstly, $x \leq y$ if and only if $X(x) \subset X(y)$. Hence $X(w)^T = \{x \leq w\}$. Note that we will usually use $[x, w]$ to denote $\{x \leq w\}$. Secondly, if $\ell(w)$ denotes the length of $w \in W$, then $\ell(w) = \dim X(w)$.

For simplicity, let X denote $X(w)$. The set $E(X, x)$ of T-invariant curves, or briefly, T-curves, in X containing the T-fixed point x turns out to be of basic importance in determining the singular locus of X (cf. [2, 3]). Let $\Phi \subset X(T)$ be the root system of (G,T), and recall that to each $\alpha \in \Phi$, there is a one dimensional unipotent subgroup U_α of G called the *root subgroup* associated to α. Recall that the positive roots Φ^+ can be described as those such that $U_\alpha \subset B$. Then any $C \in E(X, x)$ has the form $\overline{U_\alpha x}$ for some α. Moreover, $C^T = \{x, y\}$, where $y = r_\alpha x$, r_α denoting the reflection corresponding to α. When $y > x$, then $\alpha < 0$ and we can write $C = \overline{U_\beta y}$ with $\beta = -\alpha > 0$, so one can translate the Zariski tangent space $T_y(X)$ to X at y along $C\backslash\{x\}$ via U_β leaving X invariant. Taking the limit gives a T-stable subspace $\tau_C(X, x)$ of $T_x(X)$ of dimension $\dim T_y(X)$. The key result is

Peterson's Theorem *Suppose that $X = X(w)$ is nonsingular at all $y \in W$ such that $x < y \leq w$ and that all $\tau_C(X,x)$ coincide when $C \in E(X,x)$ has the property that X is nonsingular on $C\backslash\{x\}$. Then X is nonsingular at x.*

The idea of Peterson's proof is to show that if all the $\tau_C(X,x)$ coincide, then the fibre over x in the Nash blow up of X at x contains no T-curves. Since Schubert varieties are normal, it follows from Zariski's Connectedness Theorem and Lemma 2.2 that this fibre consists of a single point. This implies, by a result of Nobile [15], that X is nonsingular at x. Using this, Peterson was able to show

Theorem 1.1. *If G is of type ADE, then every rationally smooth point of a Schubert variety $X(w)$ in G/B is nonsingular.*

Combining this result with the characterizations of rationally smooth Schubert varieties given in [2], we get several lovely descriptions of the nonsingular Schubert varieties in G/B for the simply laced setting.

Theorem 1.2. (cf Theorem A of [2]) *Let G be semi-simple. Then a Schubert variety $X(w)$ in G/B is rationally smooth if and only if any of the following equivalent conditions hold:*

(1) *the Poincaré polynomial of $X(w)$*
$$P(X(w),t) = \sum b_i(X(w))t^i = \sum_{x \leq w} t^{2\ell(x)}$$
is symmetric;

(2) *the order $\leq$ on $[e,w]$ is rank symmetric;*

(3) *for each $x \leq w$, $|E(X(w),x)| = \ell(w)$; and*

(4) *the average $a(w)$ of the length function on $[e,w]$ is $\frac{1}{2}\ell(w)$. that is,*
$$a(w) = \frac{1}{|[e,w]|}\sum_{x \leq w} \ell(x) = \frac{1}{2}\ell(w).$$

We therefore obtain

Corollary 1.3. *If G is simply laced, then a Schubert variety $X(w)$ in G/B is nonsingular if and only if any of the equivalent conditions* (1)-(4) *hold.*

A corresponding G/P version will be stated in §11.

We now describe some generalizations of these results for arbitrary irreducible T-stable subvarieties X of G/B proved in this paper. Put
$$TE(X,x) = \sum_{C \in E(X,x)} T_x(C).$$

If $C = \overline{U_\alpha x}$, then $T_x(C)$ is a T-stable line of weight α, so $TE(X,x)$ is a T-submodule of $T_x(X)$ such that $\dim TE(X,x) = |E(X,x)|$ (cf. [2]). In particular $\dim T_x(X) \geq |E(X,x)|$. We will call $C \in E(X,x)$ *good* if X is nonsingular along the open T-orbit in C, or, equivalently, if C is not contained in the singular locus of X. Our generalization of Peterson's Theorem goes as follows:

Theorem 1.4. *Suppose* $\dim X \geq 2$ *and* $x \in X^T$. *Then a necessary and sufficient condition that* X *be nonsingular at* x *is that there exist at least two distinct good* T*-curves* $C, D \in E(X,x)$ *such that*

$$\tau_C(X,x) = \tau_D(X,x) = TE(X,x). \tag{1}$$

If X *is Cohen-Macaulay at* x*, then* X *is nonsingular at* x *if and only if there exists at least one good* $C \in E(X,x)$ *such that* $\tau_C(X,x) = TE(X,x)$.

The proof only uses the Zariski-Nagata Theorem, and hence is completely algebraic. In particular, it works over any algebraically closed field. (This improves the proof given in [10].) If X is a Schubert variety, it is not hard to show that Theorem 1.4 implies Peterson's Theorem, giving the first of four several proofs. The Cohen-Macaulay statement is proved in Proposition 7.8.

In order to generalize Peterson's ADE Theorem, we need to understand where the $\tau_C(X,x)$ are situated in $T_x(X)$. For this, let $\Theta_x(X)$ denote the linear span of the reduced tangent cone of X at x. If $C \in E(X,x)$ has the form $C = \overline{U_\alpha x}$, we will call C *long* or *short* according to whether α is long or short. If G is simply laced, then, by convention, all T-curves will be called short. Clearly,

$$TE(X,x) \subset \Theta_x(X) \subset T_x(X).$$

The next result is one of our key observations.

Theorem 1.5. *Assume* G *has no* G_2 *factors. Then, if* $C \in E(X,x)$ *is good,*

$$\tau_C(X,x) \subset \Theta_x(X).$$

Moreover, if C *is short, then* $\tau_C(X,x) \subset TE(X,x)$*. In particular, if* G *is simply laced and* C *is good, then* $\tau_C(X,x) \subset TE(X,x)$

We give an example in §7 which shows that the G_2 hypothesis is necessary. Peterson's ADE Theorem is now a simple consequence. Indeed, assuming $X = X(w)$, it suffices to suppose x is a rationally smooth T-fixed point such that X is smooth at every y with $x < y \leq w$. Since the singular locus of a Schubert variety has codimension at least two, $\ell(x) \leq \ell(w) - 2$. Hence x lies on at least two good T-curves (cf.

Proposition 2.3). The proof now follows from Theorems 1.4 and 1.5, since if X is rationally smooth at x, then $|E(X,x)| = \dim X$.

This result now gives us a complete description of the smooth points of Schubert varieties in G/B as long as G contains no G_2 factors.

Corollary 1.6. *Suppose G contains no G_2 factor. Then the Schubert variety $X = X(w)$ is smooth at $x < w$ if and only if $\dim \Theta_y(X) = \dim X$ for all $y \in [x,w]$. In other words, X is smooth at x if and only if the reduced tangent cones of X at all $y \in [x,w]$ are linear. In particular, X is smooth if and only if all its reduced tangent cones are linear.*

The proof is essentially the same as that of the above proof of the ADE Theorem.

A natural question is whether Peterson's ADE Theorem holds for arbitrary T-varieties in G/B if G is simply laced. It turns out that the answer is in general no, but we do have the following:

Corollary 1.7. *If G is simply laced, X is rationally smooth at x and $\dim X \geq 2$, then X is smooth at x if and only if $E(X,x)$ contains at least two good T-curves.*

Indeed, if X is rationally smooth at x, then by a recent result of Brion [1], $|E(X,x)| = \dim X$. Hence the corollary follows immediately from Theorem 1.4.

If X is a Schubert variety and G is simply laced, then we know from [2, 4] that $\Theta_x(X) = TE(X,x)$. In fact, if G is simply laced, this turns out to be true for all irreducible T-subvarieties of G/B.

Theorem 1.8. *Assume G is simply laced, and $x \in X^T$. Then every T-line in the reduced tangent cone to X at x has the form $T_x(C)$ for some $C \in E(X,x)$. That is,*

$$\Theta_x(X) = TE(X,x).$$

We now briefly describe the rest of the paper. First of all, in §3, we define the Peterson map in a general setting and derive its basic properties. In particular, if X is a Schubert variety, we show there is a remarkable explicit formula for $\tau_C(X,x)$ for any (not necessarily good) $C \in E(X,x)$. In §6, we prove a fundamental lemma showing that the Peterson map for a good T-curve C is completely determined by its behavior on the T-surfaces in X containing C. This gives us Theorem 1.5 and, in addition, allows us to deduce that certain weights outside $TE(X,x)$ may occur in $\Theta_x(X)$. However, the fact that there is no general description of $\Theta_x(X)$ makes it desirable to find a subspace containing $\tau_C(X,x)$, assuming C is good, admitting an explicit

description. We describe such a subspace in §8. In the next section, we mention an algorithm for finding the singular locus of a Schubert variety in G/B. In §11, we prove a lemma which extends our results to any G/P, with the suitable restrictions on G, and in the last section we mention some open problems.

A remark about the field is in order. Although we are assuming $k = \mathbb{C}$, we believe our arguments are valid over any algebraically closed field. This goes hand in hand with the fact proved in [16] that the singular locus of a Schubert variety is independent of the field of definition.

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2. Preliminaries on T-varieties

Throughout this paper, T will denote an algebraic torus over $k = \mathbb{C}$ with character group $X(T)$ and dual group $Y(T)$ of one parameter subgoups of T. X will always denote an irreducible T-variety with finite non-empty fixed point set X^T which is locally linearizable in the following sense: every point $z \in X$ has a connected affine T-stable neighborhood X_z admitting a T-equivariant embedding into an affine space V with a linear T-action. This is for example true for closed T-stable subsets of a normal T-variety ([17],[18]). For any T-variety X and $x \in X$ we choose once and for all such a neighborhood X_x.

We will denote the set of weights of a T-module V by $\Omega(V)$. If $x \in X^T$ and all elements of $\Omega(T_x(X))$ lie on one side of a hyperplane in $X(T) \otimes \mathbb{Q}$, then x is called *attractive*. It follows immediately from the definition, that if x is attractive, there exists a one parameter group $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle > 0$, where $\langle \cdot, \cdot \rangle : X(T) \times Y(T) \to \mathbb{Z}$, is the natural pairing. Equivalently, x is attractive if and only if $\lim_{t \to 0} \lambda(t) y = x$ for all $y \in X_x$. It is well known that if $x \in X^T$ is attractive, there is a closed T-equivariant immersion $X_x \subset T_x(X)$. For example, every T-fixed point in G/B is attractive. If x is attractive and $L \subset T_x(X)$ is a T-stable line, we may consider the restriction to X_x of a T-equivariant linear projection $T_x(X) \to L$. Since L is an affine line, this restriction gives rise to a T-eigenvector $f \in k[X_x]$. We say that $f \in k[X_x]$ *corresponds* to L if f is so obtained.

Another fact, that we will use below, is

Lemma 2.1. *Let X be affine and $x \in X^T$ attractive. If Y is any affine T-variety, then a T-equivariant morphism $f : X \to Y$ is finite if and only if $f^{-1}(f(x))$ is a finite set.*

As in the introduction, $E(X, x)$ will denote the set of T-curves in X containing the point $x \in X^T$. Also as above, a T-curve C is called *good* if $C^o = C \backslash C^T \subset X^*$, where X^* is the set of nonsingular points in X. This just means that $C \cap X^*$ is nonempty. The following lemma gives a very useful fact about $E(X, x)$ (cf [2]).

Lemma 2.2. *For any $x \in X^T$,*

$$|E(X, x)| \geq \dim_x X.$$

That is, the number of T-curves in X through x is at least $\dim_x X$.

If the number of T-curves in X is finite, then there is a finite graph $\Gamma(X)$, called the *Bruhat graph* of the pair (X, T), which generalizes the Bruhat graph of the Weyl group $\Gamma(W)$ (see for example [2]). The vertices of $\Gamma(X)$ are the T-fixed points, and two $x, y \in X^T$ are joined by an edge if and only if there exists a T-curve C in X such that $x, y \in C$. When X is a T-variety in G/B, where T is maximal in B, then $\Gamma(X)$ is a subgraph of $\Gamma(W)$. In particular, if $x, y \in X^T$ are joined by an edge in $\Gamma(X)$, then there exists an $r \in R$ such that $x = ry$. If X is a Schubert variety, then $\Gamma(X)$ is a full subgraph; any edge of $\Gamma(W)$ joining $x, y \in X^T$ is also an edge of $\Gamma(X)$. Notice also that the Chevalley-Bruhat order gives a natural order on the vertices of $\Gamma(X)$. In the Schubert case, Lemma 2.2 implies

Proposition 2.3. (Deodhar's Inequality [2]) If $x < w$, then there exist at least $\ell(w) - \ell(x)$ reflections $r \in W$ for which $x < rx \leq w$.

3. The Peterson Map

Let X be a T-variety, $x \in X^T$ a locally linearizable isolated fixed point, and assume $C \in E(X, x)$. In this section, we will define and study what we call the *Peterson map* $\tau_C(\ , x)$. The version we consider here is slightly more general than the tangent space deformation considered by Peterson, which was only defined in the case of Schubert varieties. In the next section, we will relate these two deformations and give an explicit computation Peterson's version.

Our Peterson map is defined on certain subspaces of the tangent space to X at an arbitrary point $z \in C^o$ of a T-curve C in X. We may suppose the T-stable neighborhood $X_x \subset V$ is embedded equivariantly into a T-module V. Let $M \subset T_z(X)$ be a k-subspace stable under the

isotropy group S of z, and let $\mathbf{M}^o = TM \subset T(X)\,|_{C^o}$ be the T-stable vector-bundle over C^0, having fibre M over z. Define $\mathbf{M}$ to be the Zariski closure of $\mathbf{M}^o$ in $T(X)$. Then, by definition, the Peterson map assigns $\tau_C(M, x) = \mathbf{M} \cap T_x(X)$ to M. If $M = T_z(X)$, then we will denote $\tau_C(M, x)$ by $\tau_C(X, x)$. Clearly, if X is nonsingular at x, then $\tau_C(X, x) = T_x(X)$.

If C is smooth, then an alternative description of $\tau_C(M, x)$ is as follows. By the properness of Grassmannians, the T-stable vector bundle $\mathbf{M}$ on $C \cap X_x$ extends to a vector bundle $\mathbf{M}$ on C such that the restriction of $\mathbf{M}$ to C^o is $\mathbf{M}^o$. Then $\tau_C(M, x) = \mathbf{M}_x$.

The main properties of the Peterson map are given in the next result. We assume the notation defined above is still in effect.

Proposition 3.1. *Suppose X, T, $x \in X^T$ and $C \in E(X, x)$ are as above, and let M be an S-stable subspace of $T_z(X)$ of dimension m. Then:*

1. *$\tau_C(M, x)$ is a T-stable subspace of $T_x(X)$ of dimension m, and, moreover, M and $\tau_C(M, x)$ are isomorphic S-modules. If $Y \supset X$ is any ambient smooth T-variety, then as elements of the Grassmannian $\mathcal{G}_m(Y)$ of m-planes in $T(Y)$, $\tau_C(M, x) = \lim_{t\to 0} \mathrm{d}\lambda(t)M$, where λ is an arbitrary one parameter subgroup of T such that $\lim_{t\to 0} \lambda(t)z = x$.*
2. *If $M = M_1 \oplus \cdots \oplus M_t$ is the S-weight decomposition of M, then $\tau_C(M, x) = \tau_C(M_1, x) \oplus \cdots \oplus \tau_C(M_t, x)$ is the S-weight decomposition of $\tau_C(M, x)$.*
3. *If N is any T-stable subspace of $\tau_C(X, x)$, then there exists an S-stable subspace $M \subset T_z(X)$ such that $\tau_C(M, x) = N$.*

Proof. It follows from the definitions that $\tau_C(M, x)$ is T-stable. That it is a subspace of the same dimension as M also follows from the properness of the Grassmannian. Moreover, it follows easily that the limit in the Grassmannian is found by closing the bundle $T(X)$ over C^o. Given all this, the first statement follows from the second, once we have proved that $\tau_C(M_i, x)$ and M_i are isomorphic as S-modules. Now S acts on M_i by a character α_i, hence on $TM_i \subset T(X)$, as well. With TM_i being dense in $\overline{TM_i}$, it follows that $sv = \alpha_i(s)v$ for all $v \in \tau_C(M_i, x), s \in S$. It is now obvious that $\tau_C(M, x)$ decomposes as stated. Thus we have 1) and 2).

For the last statement, it is enough, of course, to consider the case where N is a line having T-weight say α. By 1) and 2), there is an S-subspace M_1 of $T_z(X)$, on which S acts by the character α restricted to S, so that $\tau_C(M_1, x)$ contains N. Let $\lambda \in Y(T)$ be a regular one parameter subgroup of T so that $\lim_{t\to 0} \lambda(t)z = x$. Thus, there is an

induced surjective morphism $f : \mathbb{A}^1 \to C_x \subset X_x$. Let V be a T-module, into which X_x embeds equivariantly with $x = 0$. Then

$$\mathbf{B} = f^*(T(X_x)) = \mathbb{A}^1 \times_{C_x} T(X_x)$$

is a T-stable subvariety of $\mathbb{A}^1 \times V$, which certainly contains $\mathbf{M}' = \overline{f^*(TM_1)}$, the latter being a vector bundle over $\mathbb{A}^1$. This means in particular that $\mathbf{M}'$ is a trivial bundle. Moreover, it is easy to see, that over $\mathbb{G}_m \subset \mathbb{A}^1$, the map $(s, w) \mapsto (s, \mathrm{d}\lambda(s)w)$ is a closed S-equivariant immersion, S acting trivially on the first factor $\mathbb{A}^1$.

Summarizing, there is a global section σ of $\mathbf{M}'$ with $\sigma(1) \in M_1$ and with $0 \neq \sigma(0) \in N$, which, over $\mathbb{G}_m$ has the form

$$\sigma(s) = (s, \mathrm{d}\lambda(s)(\sum_i s^i w_i)),$$

for suitable $w_i \in M_1 \subset V$. Let $V = \bigoplus_\beta V_\beta$ be the decomposition into T-weightspaces, so that $w_i = \sum_\beta v_{i,\beta}$ with $v_{i,\beta} \in V_\beta$. Now compare weights in the expansion

$$\sum_i \mathrm{d}\lambda(s) s^i w_i = \sum_{i,\beta} s^{i+\langle\beta,\lambda\rangle} v_{i,\beta}.$$

Since σ extends to zero and $\sigma(0) \in N \setminus \{0\}$, the term on the right hand side of (2) of degree zero occurs when $i = -\langle\alpha, \lambda\rangle$. Furthermore, for every nonzero term on the right hand side, $i + \langle\beta, \lambda\rangle \geq 0$. Since λ is regular, it follows that $\tau_C(kw_i, x) = kv_{i,\alpha}$, where $i = -\langle\alpha, \lambda\rangle$. This says $\tau_C(kw_i, x) = N$, and we are done. $\square$

4. The Peterson Map for G/B

In this section, we will explicitly compute $\tau_C(X, x)$ for a Schubert variety $X = X(w)$ in G/B, where $C \in E(X, x)$ is such that $C^T = \{x, y\}$ and $y > x$. In other words, we are considering what happens when we pass from a higher vertex of the Bruhat graph along an edge to a lower vertex. Then C can be expressed in the form $C = \overline{U_\alpha y}$ with $\alpha > 0$, hence the additive group U_α acts transitively on $C \backslash \{x\}$. Thus any subspace $M \subset T_z X$, $z \in C^o$, is the U_α-translate of a unique subspace of $T_y X$. In addition, the vector bundle $\mathbf{M}$ introduced in the previous section is defined and U_α-equivariant on C. Therefore, τ_C can be viewed as defined on subpaces of $T_y(X)$. This is the map originally considered by Peterson.

Letting $S = \ker(\alpha)$, suppose $M \subset T_y(X)$ is S-stable (resp. T-stable). Then $\mathbf{M}$ is also S-equivariant (resp. T-equivariant). Thus $\tau_C(M, x)$ is an S-module (resp. T-module), and furthermore, $\tau_C(M, x)$ is also U_α-module. (This does not require that M be S-stable.) Assuming M is

S-stable, any S-weight space V of M is a direct sum of certain $\mathbf{g}_{\beta+k\alpha}$, β is fixed, $k \leq 0$ and $y^{-1}(\alpha + k\beta) < 0$ (since $\Omega(T_y(G/B)) = y^{-1}(-\Phi^+)$).

Now suppose M is an S-weight space in $T_y(X)$ of dimension ℓ. Then M and $\tau_C(M, x)$ are isomorphic as S-modules, but possibly different when viewed as T-modules. However, the T-weights of $\tau_C(M, x)$ are not hard to determine. Indeed, since M has only one S-weight, it follows that $\Omega(M)$ is contained in a single α-string in Φ. Now $T_y(G/B)$ is a $\mathbf{g}_{-\alpha}$-module, although $T_y(X)$ need not be one. In fact there exists a unique $\mathbf{g}_{-\alpha}$-submodule M^* of $T_y(G/B)$ such that $M^* \cong M$ as S-modules. That M^* exists is clear. If M isn't already a $\mathbf{g}_{-\alpha}$-module, then one way to describe M^* is as the unique $U_{-\alpha}$-fixed point on the T-curve $\overline{U_{-\alpha}M}$ in the ordinary Grassmannian $G_\ell(T_y(G/B))$. Clearly M^* is determined by the unique $\beta \in \Phi$ lying on the α-string containing $\Omega(M)$ satisfying the condition that $y^{-1}(\beta - \ell\alpha) \notin \Phi^-$, but $y^{-1}(\{\beta, \beta - \alpha, \ldots, \beta - (\ell-1)\alpha\}) \subset \Phi^-$. That is,

$$\Omega(M^*) = \{\beta, \beta - \alpha, \ldots, \beta - (\ell - 1)\alpha\}.$$

We will call β the *leading weight* of M^*.

Proposition 4.1. *Assuming M is as above, $\tau_C(M, x) = \mathrm{d}\dot{r}_\alpha(M^*)$, where $\mathrm{d}\dot{r}_\alpha$ denotes the differential at y of a representative $\dot{r}_\alpha \in N(T)$ of r_α. Consequently, $\Omega(\tau_C(M, x)) = r_\alpha\Omega(M^*)$.*

Proof. Since M^* is a $U_{-\alpha}$-module, $\dot{r}_A(M^*)$ is a U_α-module contained in $T_x(G/B)$ which is isomorphic to M as an S-module. But this condition uniquely determines $\tau_C(M, x)$. □

Remark 4.2 Of course we do not actually need to define M^* to determine $\tau_C(M, x)$. We will use this formulation in §9.

Recall that if $X(w)$ is smooth at $x \leq w$, then

$$\Omega(T_x(X(w)) = \Omega(TE(X, x)) = \{\gamma \in \Phi \mid x^{-1}(\gamma) < 0,\ r_\gamma x \leq w\}.$$

If $x < y \leq w$, then clearly $X(w)$ is smooth at y also. Using the Peterson map, we can now describe how to obtain the T-weights of $T_x(X(w))$ by degenerating to x along edges of $\Gamma(X(w))$. Denoting $\Omega(T_x(X(w)))$ by $\Phi(x, w)$ as in [2] and putting $\Phi(x, w)^* = \Omega(T_x(X(w))^*)$, Proposition 4.1 gives

Corollary 4.3. *Let $X(w)$ be smooth at two adjacent vertices of the Bruhat graph $\Gamma(X(w))$, say x and $y = rx$, where $y > x$ and $r \in R$. Then*

$$\Phi(x, w) = r(\Phi(y, w)^*).$$

Consequently, if $x < z \leq w$ is another vertex of $\Gamma(X(w))$ adjacent to x, then $r(\Phi(y, w)^*) = t(\Phi(z, w)^*)$, where $x = tz$ with $t \in R$.

We will see later that if $X(w)$ is smooth at y but not necessarily at $x = ry < y$, then it is still true that $r(\Phi(y,w)^*) \subset \Phi(x,w)$, provided the corresponding T-curve $C \in E(X,x)$ is short. Note: we are assuming (contrary to the common practice) that if Φ is simply laced, then all its elements are short. Thus all T-curves in the corresponding G/B are by convention short.

If C is long, the situation is more complicated. This is illustrated in the following example.

Example 4.4 Suppose G is of type B_2, and $w = r_\alpha r_\beta r_\alpha$, where α the short simple root and β the long simple root. In this example, we will compute the Peterson maps and use the result to determine the singular locus of $X(w)$, which is of course already well known. Put $X = X(w)$ and $\Omega(T_x(X)) = \Omega(x)$. If $x \leq w$ is a nonsingular point of X, then $\Omega(x) = \Phi(x,w)$. Clearly (for example, by Peterson's Theorem), w, $r_\alpha r_\beta$ and $r_\beta r_\alpha$ are nonsingular points, and one easily sees that

(1) $\Omega(w) = \{\alpha, \alpha+\beta, 2\alpha+\beta\}$;
(2) $\Omega(r_\alpha r_\beta) = \{\alpha, 2\alpha+\beta, -(\alpha+\beta)\}$;
(3) $\Omega(r_\beta r_\alpha) = \{-\alpha, \beta, \alpha+\beta\}$;

It remains to test whether the points r_α and r_β are nonsingular. Indeed, since α is simple and $r_\alpha w < w$, $\dot{r}_\alpha X = X$. Moreover, if $C = \overline{U_\alpha x}$, then $\Omega(\tau_C(X,x)) = r_\alpha(\Omega(r_\alpha x))$ as long as $r_\alpha x < x \leq w$. Thus e is a nonsingular point if and only if r_α is. Let's first compute $\Omega(\tau_C(X, r_\alpha)$ where $C = \overline{U_\beta r_\beta r_\alpha}$. It is clear that $\Omega(r_\beta r_\alpha)$ is the set of weights of of a $\mathbf{g}_{-\beta}$-submodule of $T_{r_\beta r_\alpha}(G/B)$, so

$$\Omega(\tau_C(X, r_\alpha)) = r_\beta(\Omega(r_\beta r_\alpha)) = \{-(\alpha+\beta), -\beta, \alpha\}.$$

Next consider $\Omega(\tau_D(X, r_\alpha))$ where $D = \overline{U_{2\alpha+\beta} r_\alpha r_\beta}$. It is again clear that $\Omega(r_\alpha r_\beta)$ is the set of weights of of a $\mathbf{g}_{-(2\alpha+\beta)}$-submodule of $T_{r_\alpha r_\beta}(G/B)$, so

$$\Omega(\tau_D(X, r_\alpha)) = r_{2\alpha+\beta}(\Omega(r_\alpha r_\beta)) = \{-(\alpha+\beta), \alpha, -(2\alpha+\beta)\}.$$

Hence X is singular at r_α. Now consider $\tau_D(X, r_\beta)$ for $D = \overline{U_\alpha r_\alpha r_\beta}$. By the previous comment,

$$\Omega(\tau_D(X, r_\beta)) = r_\alpha(\Omega(r_\alpha r_\beta)) = \{-\alpha, \beta, -(\alpha+\beta)\}.$$

It remains to compute $\Omega(\tau_C(X, r_\beta))$ for $C = U_{\alpha+\beta} r_\beta r_\alpha$. Organizing $\Omega(r_\beta r_\alpha)$ into $-(\alpha+\beta)$-strings gives

$$\Omega(r_\beta r_\alpha) = \{-\alpha, \beta\} \cup \{\alpha+\beta\}.$$

Since $(r_\beta r_\alpha)^{-1}(2\alpha+\beta) > 0$, it follows from Proposition 4.1 that

$$\Omega(\tau_D(X, r_\beta)) = r_{\alpha+\beta}(\{-\alpha, -(2\alpha+\beta), \alpha+\beta\}) = \{-\alpha, \beta, -(\alpha+\beta)\}.$$

Thus, by Peterson's Theorem, X is nonsingular at r_β. By the remark above, $\Omega(e) = r_\alpha(\Omega(r_\alpha))$, so

$$\Omega(e) = \{-\beta, -(\alpha+\beta), -\alpha, -(2\alpha+\beta)\}.$$

The upshot of this calculation is that the singular locus of $X(w)$ is $X(r_\alpha)$.

5. A Criterion For Smoothness Of T-varieties

In this section we will prove a generalization of Theorem 1.4. Let X be an irreducible T-variety, and let $x \in X^T$ be an attractive T-fixed point. Since the action of T is linearizable, and since smoothness is a local property we may assume that $X = X_x$. Note that we are not assuming here that $E(X, x)$ is finite.

Lemma 5.1. *Let $f : X \to Y$ be a quasi-finite equivariant morphism of T-varieties with Y nonsingular at $f(x)$. Let $Z \subset X$ be the ramification locus of f, i.e. the closed subvariety of points, at which f is not étale. Then either Z equals X, Z is empty or Z has codimension one at x.*

Proof. Assume that $\operatorname{codim}_x Z \geq 2$. We have to show that Z is empty. First of all, since x is attractive, the image of f is contained in every connected open T-stable affine neighborhood of $f(x)$, hence in $Y_{f(x)}$. Viewing f as a map to $Y_{f(x)}$, the fibre of f over $f(x)$ is finite, hence f is finite. Thus, $f(X)$ is a closed subset of the unique irreducible component of $Y_{f(x)}$ through $f(x)$. Since f is smooth somewhere it follows that $\dim X = \dim_x Y_{f(x)}$, so $f(X)$ is the unique component of $Y_{f(x)}$ through $f(x)$. It follows that $f(x)$ is an attractive fixed point of $Y_{f(x)}$, and therefore $Y_{f(x)}$ is nonsingular. Passing to the normalization $\tilde{X}$ of X, we obtain an equivariant finite map $\tilde{f} : \tilde{X} \to Y_{f(x)}$, which is étale in codimension one, because the natural map $\tilde{X} \to X$ is clearly an isomorphism over $X \setminus Z$. Thus, by the theorem of Zariski-Nagata [7], $\tilde{f}$ is étale everywhere. Hence for some point $\tilde{x} \in \tilde{X}$ which maps to $x \in X$ we have $T_{\tilde{x}}(\tilde{X}) \cong T_{f(x)}(Y_{f(x)})$ via $\mathrm{d}f$. This implies that $\tilde{X}$ is attractive, forcing $\tilde{f}$ to be an isomorphism. Thus f is birational. But being finite, f is also an isomorphism, so we are through. $\square$

This gives the following criterion for smoothness of attractive T-actions.

Theorem 5.2. *Let X be as above and let x be an attractive fixed point. Suppose there is a subset $E \subset E(X, x)$ such that every $C \in E$ is good which satisfies the following conditions:*

(1) $|E(X, x) \setminus E| \leq \dim X - 2$.

(2) $\tau_C(X,x) = \tau_D(X,x)$ *for all* T*-curves* $C, D \in E$.
(3) *If* $\tau(E)$ *denotes the common value of* $\tau_C(X,x)$ *for* $C \in E$, *then* $T_x(C) \cap \tau(E) \neq 0$ *for all curves* $C \in E(X,x)$.

Then x *is a nonsingular point of* X.

Proof. Since x is attractive we may assume that $X \subset T_x(X)$ and $x = 0$. Fix an equivariant projection $\tilde{p} : T_x(X) \to \tau(E)$, and denote its restriction to X by p. Since $T_x(C) \cap \tau(E) \neq 0$ for all curves $C \in E(X,x)$, it follows that there is no T-curve in $p^{-1}(0)$, so by Lemma 2.2, $\dim p^{-1}(0) = 0$. This implies p is finite, since x is attractive. Let Z be the ramification locus of p. According to Lemma 5.1, we are done if $\mathrm{codim}_x Z \geq 2$. By assumption, if $C \in E$, then $C^o \subset X^*$. It follows that $C^o \subset Z$ if and only if dp has a nontrivial kernel $L \subset T_z(X)$ for some $z \in C^o$. But then $\tau_C(L) \subset \ker \mathrm{d}p \cap \tau(E)$. With p being the projection to $\tau(E)$, the latter is trivial, so $\tau_C(L)$ and hence L both are equal to 0. We conclude that $E \cap E(Z,x)$ is empty. Thus, by condition 1), $|E(Z,x)| \leq \dim X - 2$ forcing $\dim_x Z \leq \dim X - 2$, thanks again to Lemma 2.2. This ends the proof. $\square$

Remark 5.3 Note that the last condition is automatically satisfied for curves $C \in E$ since $\tau_C(C,x) \subset \tau(E)$ for such a curve. Moreover, if all curves in $E(X,x)$ are smooth and have non collinear weights, then the last condition is equivalent to saying that $\tau(E) = TE(X,x)$. This in turn implies that E consists of good curves if $E \subset E(X,x)$ is a set satisfying 2) and $|E(X,x)| = \dim X$.

We immediately conclude the following corollary, which implies the first part of Theorem 1.4.

Corollary 5.4. *Suppose that either* $|E(X,x)| = \dim X$ *or all* $C \in E(X,x)$ *are nonsingular and any two distinct* $C, D \in E(X,x)$ *have distinct tangents. Suppose also that there exist two distinct good* T*-curves* $C, D \in E(X,x)$ *such that*

$$\tau_C(X,x) = \tau_D(X,x) = TE(X,x). \tag{2}$$

Then X *is nonsingular at* x.

We will prove the Cohen-Macaulay assertion in Proposition 7.8.

Remark 5.5 If X is normal, one does not need to assume x is attractive since the Zariski-Nagata Theorem can be directly applied.

Remark 5.6 If X is a H-variety for some algebraic group H, and (S,T) is an attractive slice to a H-orbit Hx (i.e. $S \subset X$ is locally closed, affine, stable under some nontrivial torus $T \subset H_x$, such that x

is an isolated point of $S \cap Hx$ and the natural mapping $H \times S \to X$ is smooth at x), then $T_x(Hx) \subset \tau_C(X,x)$. More precisely, one has $\tau_C(X,x) = \tau_C(S,x) \oplus T_x(Hx)$ for all $C \in E(S,x) \subset E(X,x)$. Thus the third condition in the theorem is always satisfied if $E(X,x) = E(S,x) \cup E(Hx,x)$ and $E = E(S,x)$.

If X is a Schubert variety $X(w)$, an explicit attractive slice for X at any $x \leq w$ is given as follows.

Lemma 5.7. *Let U be the maximal unipotent subgroup of B and U^- the opposite maximal unipotent subgroup, and suppose $x < w$. Then an attractive slice for $X(w)$ is given by the natural multiplication map*

$$(U \cap xU^-x^{-1}) \times X(w) \cap U^-x \rightarrow X(w).$$

We can now give a proof of Peterson's Theorem (cf. §1). If $x < w$ and $\ell(w) - \ell(x) = 1$, there is nothing is to prove, since Schubert varieties are nonsingular in codimension one. Letting E be the set of $C \in E(X,x)$ such that $C^T \subset [x,w]$, the existence of a slice and the hypothesis of Peterson's Theorem imply that conditions 2) and 3) of Theorem 5.2 hold. If $\ell(w) - \ell(x) \geq 2$, then Deodhar's inequality (Proposition 2.3) implies 1) holds. Hence X is nonsingular at x.

6. A Fundamental Lemma

In this section, X will denote a T-variety. We will now prove a basic lemma which allows us to deduce good properties of the Peterson translate from good properties of the Peterson translates $\tau_C(\Sigma, x)$, where Σ ranges over the T-stable surfaces containing a good $C \in E(X,x)$.

Lemma 6.1. *If $C \in E(X,x)$ is a good curve we have*

$$\tau_C(X,x) = \sum_{\Sigma} \tau_C(\Sigma, x)$$

where the sum ranges over all T-stable irreducible surfaces Σ containing C.

Proof. Let $L \subset \tau_C(X,x)$ be a T-stable line. Then by Proposition 3.1 there is an S-line $M \subset T_z(C)$, where S is the isotropy group of an arbitrary $z \in C^o$, such that $\tau_C(M,x) = L$. As X is nonsingular at z, there exists an S-stable curve D satisfying $M \subset T_z(D)$. Setting $\Sigma = \overline{TD}$ we obtain a T-stable surface, which contains C, and which satisfies $L \subset \tau_C(\Sigma, x)$. ∎

Although the lemma is almost obvious, it is a great help in the case when X is a T-stable subvariety of G/B, where G has no G_2-factors. One reason for this is

Proposition 6.2. *Suppose G has no G_2-factors and let Σ be an irreducible T-stable surface in G/B. Let $\sigma \in \Sigma^T$. Then $|E(\Sigma, \sigma)| = 2$, and either Σ is nonsingular at σ or the weights of the two T-curves to Σ at σ are orthogonal long roots α, β in B_2. In this case, Σ_σ is isomorphic to a surface of the form $z^2 = xy$ where $x, y, z \in k[\Sigma_\sigma]$ have weights $-\alpha, -\beta, -1/2(\alpha + \beta)$ respectively. In particular, if G is simply laced, then Σ is nonsingular.*

Proof. The first claim follows easily from the fact that Σ has a dense two dimensional T-orbit (cf [5]). Let C, D denote the two elements of $E(\Sigma, \sigma)$, and let α, β denote their weights. For any function $f \in k[\Sigma_\sigma]$ of weight ω corresponding to a T-line L in $T_x(\Sigma)$, there is a positive integer N such that $N(-\omega) \in \mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta$. Note that the functions corresponding to C and D have weights $-\alpha$ and $-\beta$ respectively (thus, the minus sign for ω). Except for the case where α, β and $-\omega$ are contained in a copy of $B_2 \subset \Phi$ this actually implies that $-\omega = a\alpha + b\beta$ for suitable nonnegative integers a, b. Using the multiplicity freeness of the representation of T on $k[\Sigma_\sigma]$, one is done in these cases. In the remaining case, it turns out that Σ is nonsingular at σ unless α, β are orthogonal long roots in B_2 ([5]). Let $\gamma = 1/2(\alpha + \beta)$. Then Σ_σ isomorphic to $z^2 = xy$ where $x, y, z \in k[\Sigma_\sigma]$ correspond to T-lines in $T_\sigma(\Sigma)$ of weights α, β and γ. ☐

7. The Span Of The Tangent Cone Of A T-Variety In G/B

In this section and for the rest of this paper we will assume that X is a closed irreducible T-stable subvariety of G/B and that the underlying semi-simple group G has no G_2 factors. Recall that all T-curves in G/B are smooth, and two distinct T-curves have different weights. Moreover, if $C \in E(X, x)$, then $T_x(C) = \mathcal{T}_x(C) \subset \mathcal{T}_x(X)$. In particular, $TE(X, x) \subset \Theta_x(X)$, the k-linear span of the reduced tangent cone $\mathcal{T}_x(X)$ of X at x.

We will now study the Peterson translates $\tau_C(X, x)$ of X along good T-curves C in X, where $x \in C^T$. We will first show that each $\tau_C(X, x)$ is a subspace of $\Theta_x(X)$. Hence the tangent spaces at T-fixed points behave well under the Peterson map. On the other hand, it is an open question as to how to explicitly describe $\Theta_x(X)$ for any $x \in X^T$. The only relevant fact we know of is the following, proved in [4].

Proposition 7.1. *Let S be an algebraic torus over k and V a finite S-module. Suppose Y is a Zariski closed S-stable cone in V, and let $\mathcal{H}(Y)$ denote the convex hull of*

$$\Phi(Y) = \{\alpha \in X(S) \mid V_\alpha \subset Y\}$$

in $X(S)\otimes\mathbb{R}$, where V_α denotes the α-weight space in V. Also let $\Theta(Y)$ denote the k-linear span of Y in V. Then

$$\Omega(\Theta(Y)) \subset \mathcal{H}(Y).$$

In particular, if Φ is simply laced, then $\Omega(\Theta_x(X)) = \Phi \cap \mathcal{H}(\mathcal{T}_x(X))$, so $\Theta_x(X)$ is the k-linear span of the set of T-lines in $\mathcal{T}_x(X)$. In §9, we will show that if G is simply laced, then, in general, $\Theta_x(X) = TE(X,x)$. We now prove of one of our main results.

Theorem 7.2. *Suppose that X is an arbitrary T-stable subvariety of G/B. Then for any $x \in X^T$ and any good curve $C \in E(X,x)$ we have*

$$\tau_C(X,x) \subset \Theta_x(X).$$

Moreover, if C is short, then $\tau_C(X,x) \subset TE(X,x)$.

Proof. Since $\tau_C(X,x)$ is generated by T-invariant surfaces and since $\Theta_x(\Sigma) \subset \Theta_x(X)$ for all surfaces $\Sigma \subset X$ which contain x, it is enough to show the proposition when X is a surface. If X is nonsingular at x, then $\tau_C(X,x) = T_x(X)$. Otherwise we know from Proposition 6.2 that X is a cone over x, and for a cone, $\Theta_x(X) = T_x(X)$. Since X is nonsingular at x when C is short, the last statement is obvious. □

The last assertion of the theorem gives us a generalization of Peterson's ADE Theorem.

Corollary 7.3. *Let G be simply laced, and suppose X is rationally smooth at x. Then X is nonsingular at x if and only if there are two good T-curves in $E(X,x)$. Moreover, if X is Cohen-Macaulay, one good T-curve suffices.*

Proof. By a result of Brion [1], if X is rationally smooth at x, then $|E(X,x)| = \dim X$. By Theorem 7.2, we have $\tau_C(X,x) = TE(X,x)$ for every good $C \in E(X,x)$. Hence the result follows from Theorem 1.4. We will prove the last statement below. □

Remark 7.4 In general, Theorem 7.2 does not hold if $G = G_2$. For example, consider the surface Σ given by $z^2 = xy^3$ in $\mathbb{A}^3$. Let α, β, γ be characters of T, satisfying $\gamma = 2\alpha + 3\beta$, and let T act on $\mathbb{A}^3$ by

$$t \cdot (x,y,z) = (t^\alpha x, t^{(\alpha+2\beta)} y, t^\gamma z).$$

Clearly Σ is T-stable, and its reduced tangent cone at 0 is by definition $\ker \mathrm{d}z$, hence is linear. The T-curve $C = \{x = 0\}$ is good, and along C^o, we have $T_v(\Sigma) = \ker \mathrm{d}x$. It follows that $\tau_C(\Sigma, x) = \ker \mathrm{d}x$, which is not a subspace of $\Theta_0(\Sigma)$. It remains to remark that Σ is open in a T-stable surface in G_2/B, where T is the usual maximal torus and α and β are respectively the corresponding long and short simple roots.

We now generalize Peterson's ADE Theorem in a different direction. That is, we study which rationally smooth T-fixed points of a Schubert variety (or more generally, a T-variety in G/B) are nonsingular without the assumption G is simply laced. Since $\dim_x \mathcal{T}_x(X) = \dim X$, it is obvious that $\mathcal{T}_x(X)$ is linear if and only if $\dim_k \Theta_x(X) = \dim X$. Thus, as a consequence of Theorems 7.2 and 1.4, we obtain

Theorem 7.5. *A T-variety $X \subset G/B$ is nonsingular at $x \in X^T$ if and only if $\Theta_x(X)$ has minimal dimension $\dim X$ and $E(X,x)$ contains at least two good curves.*

Specializing to the Schubert case gives

Corollary 7.6. *A Schubert variety $X = X(w)$ is nonsingular at $x \in X^T$ if and only if all reduced tangent cones $\mathcal{T}_y(X)$, $x \le y \le w$, are linear. Consequently, the nonsingular Schubert varieties are exactly those whose tangent cones are linear at every T-fixed point.*

The proof is similar to the proof of Peterson's Theorem given at the end of §5 so we will omit it.

Remark 7.7 Corollary 7.6 gives another proof of Peterson's ADE Theorem since if G is simply laced and $X = X(w)$ is rationally smooth at x, then its tangent cones are linear at every T-fixed point y with $x \le y \le w$ [2].

We now prove the second assertion of Theorem 1.4 that if X is Cohen-Macaulay one good T-curve such that $\tau_C(X,x) = TE(X,x)$ suffices to guarantees that x is nonsingular.

Proposition 7.8. *Suppose X is Cohen-Macaulay and $x \in X^T$. X is nonsingular at x if and only if there is a good T-curve C in $E(X,x)$ with $\tau_C(X,x) = TE(X,x)$.*

Proof. Let $x_1, \dots, x_n \in k[X_x]$ be the functions corresponding to the T-curves $C_1, \dots, C_n$ in $E(X_x, x)$. Since $\dim \tau_C(X,x) = \dim X$, and since the T-curves are smooth and have non collinear weights, n equals the dimension of X. We may assume that $C = C_1$.

Let $\mathbf{a} \subset k[X_x]$ be the ideal generated by $x_2, \dots, x_n$. Then $\mathbf{a}$ is contained in the ideal of C. Now $\tau_C(X_x, x)$ equals the span $TE(X,x)$ of the T-curves C_i. This means that the differentials $\mathrm{d}x_i$ are independent along C^o. As $C \cong \mathbb{A}^1$ is nonsingular, we are done if $\mathbf{a}$ is the ideal $I(C)$ of C. We know that $\mathbf{a}_y = I(C)_y$ at every stalk $k[X_x]_y$ for $y \in C^o$, since the $(\mathrm{d}x_i)_y$ are independent.

As a subset of X_x, C is equal to the support of the Cohen-Macaulay subscheme $Z = \mathrm{Spec}(A/\mathbf{a})$ of X_x. Under the natural restriction, a

function f on X_x, which vanishes on C, defines a global section of $\mathcal{O}_Z$ with support contained in $\{x\}$. It is well known that on a Cohen-Macaulay scheme, the only such section is zero. So f restricts to zero, and we are done. $\square$

Corollary 7.9. *Suppose X is Cohen-Macaulay and there exists a short good $C \in E(X,x)$. Then if $|E(X,x)| = \dim X$, then X is nonsingular at x. In particular, if X is rationally smooth on C, then it is smooth on C.*

Proof. The only facts we have to note are, firstly, that if C is short, then $\tau_C(X,x) \subset TE(X,x)$ and, secondly, if X is rationally smooth at $x \in X^T$, then $|E(X,x)| = \dim X$ (see [1]). $\square$

Remark 7.10 Since Schubert varieties are Cohen-Macaulay, we obtain from this yet another proof of Peterson's ADE Theorem.

For locally complete intersections or normal T-orbit closures in G/B, Proposition 7.8 puts quite strong restrictions onto the Bruhat graph $\Gamma(X)$. In fact, recalling that X^* denotes the set of nonsingular points of X, we see that if X is Cohen-Macaulay, then every rationally smooth vertex of $\Gamma(X)$ connected to a vertex of $\Gamma(X^*)$ is in fact a vertex of $\Gamma(X^*)$. Therefore we get the following

Corollary 7.11. *Suppose G is simply laced and X is Cohen-Macaulay , rationally smooth and $\Gamma(X^*)$ is non-trivial. Then X is nonsingular as long as $\Gamma(X)$ is connected.*

Remark 7.12 In fact, if X is nonsingular, then its Bruhat graph is connected.This can be shown by considering the Bialynicki-Birula decomposition of X induced by a regular element of $Y(T)$. We will omit the details.

8. More On Peterson Translates and Schubert Varieties

The purpose of this section is to adress the problem that there is in general no nice description of $\Theta_x(X)$. Hence we would like to find a more precise picture of $\tau_C(X,x)$ when X is a Schubert variety in G/B, say $X = X(w)$ and, as usual, G is not allowed any G_2 factors. Of course, if G is simply laced or C is short, we've already shown $\tau_C(X,x) \subset TE(X,x)$.

It turns out that if C is long, there is a T-subspace of $\Theta_x(X)$, depending only on $TE(X,x)$ and the isotropy group B_x of x in B, which contains most of $\tau_C(X,x)$, and the part that fails to lie in this subspace is

easy to describe. Let $\mathbf{g}(x)$ denote the Lie-algebra of B_x, $U(\mathbf{g}(x))$ its universal enveloping algebra and define $\mathbb{T}_x(X)$ to be the $\mathbf{g}(x)$-submodule

$$\mathbb{T}_x(X) = U(\mathbf{g}(x))TE(X,x) \subset \Theta_x(X)$$

We will show that if $C \in E(X,x)$ is both good and long and $C^T \subset]x,w]$, then $\mathbb{T}_x(X)$ almost contains $\tau_C(X,x)$. In fact, taking $x = r_\alpha$ in Example 4.4 shows that in general $\tau_C(X,x) \not\subset \mathbb{T}_x(X)$. Consequently, $\Theta_x(X)$ is not in general equal to $\mathbb{T}_x(X)$.

Proposition 8.1. *Assume $C \in E(X,x)$ is long, say $C = \overline{U_{-\mu}x}$ where μ is positive and long, and suppose X is nonsingular at $y = r_\mu x$. Assume $\mathbf{g}_\gamma \subset \tau_C(X,x)$, but $\mathbf{g}_\gamma \not\subset \mathbb{T}_x(X)$, and put $\delta = \gamma + \mu$. Then:*

(1) *there exists a long positive root ϕ orthogonal to μ such that $\mathbf{g}_{-\phi} \subset TE(X,x)$ and*
$$\gamma = -1/2(\phi + \mu);$$
(2) $\mathbf{g}_{-\phi} \not\subset T_y(X)$*;*
(3) *if $\delta > 0$, then $x^{-1}(\delta) < 0$ and*
$$\mathbf{g}_\gamma \oplus \mathbf{g}_\delta \oplus \mathbf{g}_{-\mu} \subset \tau_C(X,x), \quad \mathbf{g}_\gamma \oplus \mathbf{g}_\delta \oplus \mathbf{g}_\mu \subset T_y(X);$$
(4) *on the other hand, if $\delta < 0$, then $x^{-1}(\delta) > 0$ and*
$$\mathbf{g}_\gamma \oplus \mathbf{g}_{-\delta} \oplus \mathbf{g}_{-\mu} \subset \tau_C(X,x), \quad \mathbf{g}_{-\gamma} \oplus \mathbf{g}_\delta \oplus \mathbf{g}_\mu \subset T_y(X);$$
(5) *In particular, if $D = \overline{U_{-\phi}x}$, then $\tau_C(X,x) \neq \tau_D(X,x)$.*

Proof. The existence of a long root ϕ satisfying all the conditions in part (1) except possibly positivity follows from the Fundamental Lemma (6.1), Proposition 6.2 and the fact that $\mathbf{g}_{-\mu} \subset TE(X,x)$. To see ϕ is positive suppose otherwise. It's then clear that $\delta \in \Phi^+$. If $x^{-1}(\delta) > 0$ also, then $\mathbf{g}_\gamma \subset \mathbb{T}_x(X)$ since $\gamma = -\mu + \delta$, contradicting the assumption. Hence $x^{-1}(\delta) < 0$. But as $\tau_C(X,x)$ is a $\mathbf{g}_\mu$-module, it follows immediately that $\mathbf{g}_\gamma \oplus \mathbf{g}_\delta \subset \tau_C(X,x)$. Since μ is long, Proposition 4.1 implies

$$\mathbf{g}_\gamma \oplus \mathbf{g}_\delta \subset T_y(X).$$

Moreover, since $x^{-1}(\phi) = y^{-1}(\phi) > 0$, it also follows that $\mathbf{g}_{-\phi} \subset T_y(X)$, so $\mu,\ \delta,\ -\phi$ constitute a complete γ-string occuring in $\Omega(T_y(X))$. Since X is nonsingular at y and $\gamma, y^{-1}(\gamma) < 0$, we get the inequality $y < r_\gamma y \leq w$. Thus X is nonsingular at $r_\gamma y$. Letting E be the good T-curve in X such that $E^T = \{y, r_\gamma y\}$, we have $\tau_E(X,y) = T_y(X)$, so the string $\mu,\ \delta,\ -\phi$ also has to occur in $\Omega(T_{r_\gamma y}(X))$. In particular, $\mathbf{g}_{-\phi} \subset TE(X, r_\gamma y) = T_{r_\gamma y}(X)$, and hence $r_\phi r_\gamma y \leq w$. But this means

$$r_\gamma x = r_\gamma r_\mu y = r_\gamma r_\mu r_\gamma r_\gamma y = r_\phi r_\gamma y \leq w,$$

so $\mathbf{g}_\gamma \subset TE(X,x)$. This is a contradiction, so $\phi > 0$.

We next prove (2). Recall $y = r_\mu x$ and suppose to the contrary that $\mathbf{g}_{-\phi} \subset T_{r_\mu x}(X)$. If $\delta > 0$, we can argue exactly as above, so we are reduced to assuming $\delta < 0$. If $x^{-1}(\delta) < 0$, then $\mathbf{g}_\gamma \subset \mathbb{T}_x(X)$ due to the fact that $\gamma = -(\phi + \delta)$. Thus $x^{-1}(\delta) > 0$, whence $y^{-1}(\gamma) > 0$ and $\mathbf{g}_{-\gamma} \subset T_y(X)$ since $-\gamma > 0$. Moreover, $\mathbf{g}_\delta \subset T_y(X)$ since $\mathbf{g}_\gamma \subset \tau_C(X, x)$. Hence, we have

$$\mathbf{g}_\mu \oplus \mathbf{g}_{-\phi} \oplus \mathbf{g}_\delta \oplus \mathbf{g}_{-\gamma} \subset T_y(X).$$

Now δ and $-\gamma$ form a ϕ-string in $\Omega(T_y(X))$, and since $y < r_\phi y \leq w$, it follows as above that $\mathbf{g}_\delta \subset T_{r_\phi y}(X)$. Therefore, $r_\delta r_\phi y \leq w$. But $r_\gamma x = r_\delta r_\phi y$, so we have a contradiction. Hence, $\mathbf{g}_{-\phi} \not\subset T_{r_\mu x}(X)$. To prove (3), note that, as usual, $x^{-1}(\delta) < 0$, so $\mathbf{g}_\delta \subset \tau_C(X, x)$ by the $\mathbf{g}_\mu$-module property. To obtain (4), note that if $\delta < 0$, then $x^{-1}(\delta) > 0$, so $y^{-1}(-\gamma) < 0$. As $\gamma < 0$, this implies that $\mathbf{g}_{-\gamma} \subset T_y(X)$, so in fact $\mathbf{g}_{-\gamma} \oplus \mathbf{g}_\delta \oplus \mathbf{g}_\mu \subset T_y(X)$. The proof of (5) is clear, so the proof is now finished. $\square$

Now fix C and $\mu \in \Phi^+$ as above and let $I_\mu \subset \Phi$ consist of all negative γ such that:

(a) $\gamma = -1/2(\mu + \phi)$, where ϕ satisfies conditions (1) and (2) of Proposition 8.1,
(b) $\delta = \mu + \gamma \in \Phi$, and
(c) δ satisfies conditions (4) and (5).

Put $V_C = \bigoplus_{\gamma \in I_\mu} \mathbf{g}_\gamma$. Notice that $V_C \subset T_x(G/B)$. Proposition 8.1 thus gives the following:

Corollary 8.2. *Assuming the previous hypotheses, we have*

$$\tau_C(X, x) \subset \mathbb{T}_x(X) + V_C \subset \Theta_x(X).$$

9. The Simply Laced Case

The crucial point in the proof of Peterson's ADE Theorem is the fact that every T-stable line in the span of the tangent cone of X comes from a T-curve in X. It turns out that this is true for any closed T-variety $X \subset G/B$ as long as G is simply laced. We now prove this fact.

Theorem 9.1. *Suppose G has no G_2-factors. Let $L \subset \Theta_x(X)$ be a T-stable line with weight ω. Then*

$$\omega = \frac{1}{2}(\alpha + \beta)$$

where α and β are the weights of suitable T-curves C and D, respectively. If, moreover, G is simply laced, then $\alpha = \beta = \omega$, hence L is the tangent line of a T-curve $C \in E(X, x)$.

Proof. We will prove the following equivalent 'dual' statement: if ω is the weight of a function corresponding to a T-stable line $L \subset \Theta_x(X)$, then there are α and β with $\omega = 1/2(\alpha + \beta)$ where α and β are the weights of functions corresponding to T-curves C and D, respectively.

Let $z \in k[X_x]$ be the T-eigenfunction corresponding to L, and let

$$x_1, x_2, \ldots, x_n \in k[X_x]$$

be those corresponding to the T-curves $C_1, C_2, \ldots, C_n$ through x. Consider the unique linear projections

$$\tilde{x}_i : T_x(X) \to T_x(C_i), \quad \tilde{z} : T_x(X) \to L$$

which restrict respectively to $x_i, z \in k[X_x]$.

Since the (restriction of the) projection $X_x \to \bigoplus T_x(C)$ has a finite fibre over 0, $k[X_x]$ is a finite $k[x_1, x_2, \ldots, x_n]$-module. In particular $z \in k[X_x]$ is integral over $k[x_1, \ldots, x_n]$. We obtain a relation

$$z^N = p_{N-1} z^{N-1} + p_{N-2} z^{N-2} + \cdots + p_1 z + p_0 \tag{3}$$

where N is a suitable integer and the $p_i \in k[x_1, \ldots, x_n]$. Without loss of generality we may assume that every summand on the right hand side is a T-eigenvector with weight $N\omega$. Let $P_i \in k[\tilde{x}_1, \ldots, \tilde{x}_n]$ be polynomials restricting to p_i, having the same weight $(N - i)\omega$ as p_i. Then every monomial m of P_i has this weight too. If for all i every such monomial m has degree $\deg m > N - i$, then $p_i z^{N-i}$ is an element of $\mathbf{m}_x^{N+1}$, where $\mathbf{m}_x$ is the maximal ideal of x in $k[X_x]$. This means that $\tilde{z}$ vanishes on the tangent cone of X_x, so $L \not\subset \Theta_x(X)$, which is a contradiction.

Thus, there is an i and a monomial m of P_i, such that $\deg m \leq M = N - i$. Let $m = c\tilde{x}_1^{d_1} \tilde{x}_2^{d_2} \ldots \tilde{x}_n^{d_n}$, with integers d_j and a nonzero $c \in k$. So $\sum_j d_j \leq N$. Let α_j be the weight of $\tilde{x}_j$. Then we have

$$M\omega = \sum d_j \alpha_j$$

After choosing a new index, if necessary, we may assume that $d_j \neq 0$ for all j. Let F be a nondegenerate bilinear form on $X(T) \otimes \mathbb{Q}$ which induces the length function on Φ. We have to consider two cases. First suppose that ω is a long root, with length say L. Then $F(\alpha_j, \omega) \leq L^2$ with equality if and only if $\alpha_j = \omega$. Thus, $ML^2 = \sum d_j F(\alpha_j, \omega) \leq M \max_j F(\alpha_j, \omega) \leq ML^2$ and so there is a j with $\alpha_j = \omega$ and we are done, since this implies $\tilde{z} = \tilde{x}_j$. Note that, although we are considering

all roots short in case G is simply laced, this contains actually the case that all roots have the same length.

Now suppose ω is short, having length l. In this case $F(\alpha_j, \omega) \leq l^2$. Since $Ml^2 = MF(\omega, \omega) = \sum_j d_j F(\alpha_j, \omega)$ and since $\sum d_j \leq M$, it follows that all α_j satisfy $F(\alpha_j, \omega) = l^2$. If there is a j such that $\alpha_j = \omega$, then, as above, we are done. Otherwise for each j, α_j is long, and α_j and ω are contained in a copy $B(j) \subset \Phi$ of B_2. There is a long root $\beta_j \in B(j)$ with $\alpha_j + \beta_j = 2\omega$. We have to show that there are j_0 and j_1 so that $\beta_{j_0} = \alpha_{j_1}$. Fix $j_0 = 1$ and let $\alpha = \alpha_1$, $\beta = \beta_1$. Then $F(\alpha, \beta) = 0$. This gives us the result: $Ml^2 = MF(\omega, \beta) = 0 + \sum_{j>1} F(\alpha_j, \beta)$. Now if all $F(\alpha_j, \beta)$ are less or equal l^2, this last equation cannot hold, since $\sum_{j>1} d_j < M$. We conclude that there is a j_1 so that $F(\alpha_{j_1}, \beta) = L^2$, hence $\alpha_{j_1} = \beta$, and we are through. The statement for G simply laced follows from this, since in this case all roots have the same length, hence are long. $\square$

For completeness, we state the following corollary.

Corollary 9.2. *Suppose the G is simply laced and that X is a T-variety in G/B such that $\dim X \geq 2$. Then X is smooth at $x \in X^T$ if and only if $|E(X, x)| = \dim X$ and there are at least two good T-curves at x.*

Theorem 9.1 implies in particular that the linear spans of the tangent cones of two T-varieties behave nicely under intersections, and this allows us to deduce a somewhat surprising fact about the intersection of the tangent spaces of two T-varieties at a common nonsingular point.

Corollary 9.3. *Suppose the G is simply laced and that X and Y are T-varietes in G/B. Suppose also that $x \in X^T \cap Y^T$. Then*

$$\Theta_x(X \cap Y) = \Theta_x(X) \cap \Theta_x(Y).$$

Furthermore, if both X and Y are nonsingular at x, then

$$T_x(X \cap Y) = T_x(X) \cap T_x(Y).$$

In particular, if $|E(X \cap Y, x)| = \dim X \cap Y$, then $X \cap Y$ is nonsingular at x.

Proof. The first assertion is a consequence of Theorem 9.1 and the fact that $E(X, x) \cap E(Y, x) = E(X \cap Y, x)$. For the second, use the fact

that if a variety Z is nonsingular at z, then $T_z(Z) = \Theta_z(Z)$. Thus

$$\begin{aligned} T_x(X) \cap T_x(Y) &= \Theta_x(X) \cap \Theta_x(Y) \\ &= \Theta_x(X \cap Y) \\ &\subset T_x(X \cap Y) \\ &\subset T_x(X) \cap T_x(Y) \end{aligned}$$

The final assertion follows from the fact that if X and Y are both nonsingular at x, then $T_x(X) \cap T_x(Y) = TE(X \cap Y, x)$. □

For example, it follows that the in the simply laced setting, the intersection of a Schubert variety $X(w)$ and a dual Schubert variety $Y(v) = \overline{B^- v}$ is nonsingular at each x with $v \leq x \leq w$ as long as each of the constituents is nonsingular at x.

Remark 9.4 The previous corollary was stated in [2] (cf. Theorem H and Corollary H) for so called shifted Schubert varieties, that is any subvariety of G/B of the form $X(y, w) = yX(w)$, where $y, w \in W$. One can in fact say a little more for shifted Schubert varieties in type A because from a result of Lakshmibai and Seshadri [12] saying that if G is of type A, then $\Theta_x(X) = T_x(X)$ for every shifted Schubert variety X. Namely,

$$T_x(X) \cap T_x(Y) = T_x(X \cap Y)$$

for any two shifted Schubert varieties X and Y meeting at $x \in W$.

10. Singular Loci of Schubert Varieties

In this section we give an algorithm for computing the singular locus $X^\times$ of a Schubert variety $X = X(w)$ assuming G has no G_2 factors. Obviously $X^\times$ is a union of Schubert varieties, so we only have to compute the maximal elements of $X^{\times T}$. Schubert varieties $X = X(w)$ being Cohen-Macaulay , we know $x < w$ is a nonsingular point as long as $E(X, x)$ contains a short good T-curve and $|E(X, x)| = \dim X$. Hence maximal elements x of $X^\times$ either have the property that $|E(X, x)| > \dim X$ or every good T-curve at x is long.

On the other hand, we can use Proposition 4.1, which gives a criterion for deciding when $\tau_C(X, x) = \tau_D(X, x)$ that doesn't depend on knowing either or both of $C, D \in E(X, x)$ are good. Suppose $C = \overline{U_\alpha x}$ and $D = \overline{U_\beta z}$ where $\alpha, \beta > 0$. Let $y = r_\alpha x, z = r_\beta x$ so $y, z \in [x, w]$. By Proposition 4.1, $\tau_C(X, x) = \tau_D(X, x)$ if and only if $r_\alpha \Omega(T_y(X)^*) = r_\beta \Omega(T_z(X)^*)$, or, equivalently, $\Omega(T_y(X)^*) = r_\alpha r_\beta \Omega(T_z(X)^*)$, where $T_y(X)^*$ is the $\mathbf{g}_{-\alpha}$-submodule of $T_y(G/B)$ defined in §4. Note that

we are working in $T_y(G/B)$ instead of $T_x(X)$. Now $\mathrm{d}\dot{r_\alpha}\mathrm{d}\dot{r_\beta}(T_z(X)^*)$ is a $\mathbf{g}_{r_\alpha(\beta)}$-module, and consequently, this implies that $T_y(X)^*$ is a module for the subalgebra of $\mathbf{g}$ generated by $\mathbf{g}_{-\alpha}$ and $\mathbf{g}_{r_\alpha(\beta)}$.

Proposition 10.1. *Assuming the notation is as above, $\tau_C(X,x) = \tau_D(X,x)$ if and only if $T_y(X)^*$ is a $\mathbf{g}_{r_\alpha(\beta)}$-submodule of $T_y(G/B)$ whose leading weights are the $r_\alpha r_\beta(\gamma)$, where γ runs through the set of leading weights for the $\mathbf{g}_{-\beta}$-module $T_z(X)^*$. Moreover, if $C_1, C_2, \dots C_k \in E(X,x)$ where each $C_i^T = \{x, y_i\}$ with $y_i = t_i x > x$ and if all $\tau_{C_i}(X,x)$ coincide, then every $T_{y_i}(X)^*$ is a module for the subalgebra $\mathbf{m}_i$ of $\mathbf{g}$ generated by*

$$\mathbf{g}_{t_i(\alpha_1)} \oplus \mathbf{g}_{t_i(\alpha_2)} \oplus \cdots \oplus \mathbf{g}_{t_i(\alpha_{t_k})}.$$

The last assertion is a consequence of the Jacoby identity. Note that in this result, there is no assumption that the T-curves be good.

The algorithm for determing $X^\times$ is now clear. Suppose one knows that $y \in X^{*T}$, and assume $x = r_\alpha y < y$ where $\alpha > 0$. Clearly if

$$\mathrm{d}\dot{r_\alpha}(\Omega(T_y(X)^*)) \neq \{\gamma \mid x^{-1}(\gamma) < 0, r_\gamma x \leq w\},$$

then $x \in X^\times$. If equality holds, then it suffices to apply Proposition 10.1 to any good $D \in E(X,x)$. Thus the algorithm requires checking whether X is nonsingular at any $z \in X^{*T}$ with $z > x, z \neq y$ and $sz = x$ for some $s \in R$.

11. Generalizations to G/P

As usual, assume G is semi-simple and has no G_2 factors, and suppose P is a parabolic subgroup of G containing B. In this section, we will indicate which results extend to T-varieties in G/P. Let $\pi : G/B \to G/P$ be the natural projection. The extensions to G/P are based on the following lemma.

Lemma 11.1. *Let $Y \subset G/P$ be closed and T-stable, and put $X = \pi^{-1}(Y)$. Then:*

1. *the projection $\pi : X \to Y$ is a smooth morphism, hence $X^* = \pi^{-1}(Y^*)$;*
2. *for all $x \in X^T$, $\mathrm{d}\pi_x : T_x(X) \to T_y(Y)$ is surjective and*
$$\mathrm{d}\pi_x(\Theta_x(X)) = \Theta_y(Y),$$
where $y = \pi(x)$;
3. *$\pi(E(X,x)) = E(Y,y)$; and*
4. *if $C \in E(X,x)$ is good and $\pi(C)$ is a curve, then $\pi(C) \in E(Y,y)$ is good and*
$$\mathrm{d}\pi_x(\tau_C(X,x)) = \tau_{\pi(C)}(Y,y).$$

Proof. The first statement (1) is standard. Moreover, $\mathrm{d}\pi_x$ is surjective for all $x \in X$ and $\mathrm{d}\pi_x$ maps the schematic tangent cone of X at x onto that of Y at y. Consequently, it is also a surjection of the the associated reduced varieties. Since $\mathrm{d}\pi_x$ is linear, (2) is established. (3) is an immediate consequence of the fact that $\pi(E(G/B, x)) = E(G/P, y)$. Part (4) follows from the existence of a local T-equivariant section of π, the smoothness of π and Lemma 6.1. □

Corollary 11.2. *Assume G has no G_2 factors, and suppose Y is any T-variety in G/P. If $\dim Y \geq 2$, then Y is smooth at the T-fixed point y if and only if $E(Y, y)$ contains two good T-curves and the reduced tangent cone to Y at y is linear.*

Proof. Apply the previous lemma and Theorem 7.5 to $X = \pi^{-1}(Y)$ at any $x \in \pi^{-1}(y)^T$, which, by the Borel Fixed Point Theorem, is non-empty since $y \in Y^T$. □

If G is simply laced, there is more.

Corollary 11.3. *Assume G is simply laced. Then for any T-variety Y in G/P and $y \in Y^T$,*

$$\Theta_y(Y) = TE(Y, y).$$

In particular, if $\dim Y \geq 2$, then Y is smooth at y if and only if $|E(Y, y)| = \dim Y$ and y lies on at least two good T-curves.

We also have

Theorem 11.4. *If G is simply laced, then every rationally smooth T-fixed point of a Schubert variety Y in G/P is nonsingular.*

Proof. Let $y \in Y^T$ be a rationally smooth T-fixed point of Y. Using the relative order, we may without loss of generality assume that if $z \in Y^T$ and $z > y$, then Y is nonsingular at z. By the relative version of Deodhar's Inequality and the fact that the singular locus of Y has codimension at least two (as Y is normal), there are at least two good T-curves in $E(Y, y)$. Since $|E(Y, y)| = \dim Y$ ([1]), the proof is done. □

Finally, we state a G/P analog of Corollary 1.3.

Corollary 11.5. *If G is simply laced, a Schubert variety in G/P is nonsingular if and only if the Poincaré polynomial of Y is symmetric if and only if $|E(Y, y)| = \dim Y$ for every $y \in Y^T$.*

12. A Remark and Two Problems

Although we have not yet given an explicit example, it is definitely not true that in the simply laced setting, every rationally smooth T-variety in G/B is nonsingular. In fact, there are T-orbit closures in types D_n if $n > 4$ and in E_6, E_7, E_8 which are rationally smooth but non-normal, hence singular. For more information, see [5, 13]. A final comment is that one of the most basic open problems about Schubert varieties in our context is to describe the T-lines in the linear span of the tangent cone at a T-fixed point in the non-simply laced setting. Once this is settled, we will have a complete picture of the singular loci of all Schubert varietes. Another unsolved problem is to identify all the T-lines in the tangent space at at T-fixed point. There are results in this direction in papers of Lakshmibai and Seshadri [12], Lakshmibai [11] and Polo [16]. The natural conjecture that these tangent spaces are spanned by Peterson translates is, in light of Theorem 1.3, seems to be only true for type A.

References

[1] M Brion: *Rational smoothness and fixed points of torus actions* Transformation Groups, Vol. 4, No. 2-3, (1999) 127-156

[2] J B Carrell: *The Bruhat Graph of a Coxeter Group, a Conjecture of Deodhar, and Rational Smoothness of Schubert Varieties.* Proc. Symp. in Pure Math. bf 56, No. 2, (1994), Part 1, 53-61.

[3] J B Carrell: *On the smooth points of a Schubert variety.* Representations of Groups, Canadian Mathematical Conference Proceedings **16** (1995), 15-35.

[4] J B Carrell: *The span of the tangent cone of a Schubert variety.* Algebraic Groups and Lie Groups, Australian Math. Soc. Lecture Series **9**, Cambridge Univ. Press (1997), 51-60.

[5] J B Carrell and A Kurth: *Normality of Torus Orbit Closures on G/P*, preprint (1998).
1-25.

[6] V Deodhar: *Local Poincaré duality and nonsingularity of Schubert varieties*, Comm. in Algebra **13** (1985), 1379-1388.

[7] A Grothendieck: *Cohomologie Locale des Faisceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux*, SGA **2** (1962), X, 3.4

[8] D Kazhdan and G Lusztig: *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.

[9] D Kazhdan and G Lusztig: *Schubert varieties and Poincaré duality*, Proc. Symp. Pure Math. A.M.S. **36** (1980), 185-203.

[10] J Kuttler: *Ein Glattheitskriterium Für Attraktive Fixpunkte von Torusoperationen*, Diplomarbeit, U. of Basel (1998).

[11] V Lakshmibai: *Tangent spaces to Schubert varieties*, Math. Res. Lett. 2(1995), no. 4, 473–477.

[12] V Lakshmibai and C Seshadri: *Singular locus of a Schubert variety*, Bull. Amer. Math. Soc. (N.S.) **5** (1984), 483-493.

[13] J Morand: *Closures of torus orbits in adjoint representations of semisimple groups*, C.R.Acad. Sci. Paris Seŕ. I Math. **328**(1999), no. 3, 197-202.

[14] D Mumford: *The Red Book of Varieties and Schemes.* Springer LN **1358**, Springer-Verlag, Berlin Heidelberg (1988).

[15] A Nobile: *Some properties of the Nash blowing up*, Pac. Jour. Math. **60** (1975), 297-305.

[16] P Polo: *On Zariski tangent spaces of Schubert varieties and a proof of a conjecture of Deodhar*, Indag. Math. **11** (1994), 483-493.

[17] H Sumihiro: *Equivariant Completion*, J. Math. Kyoto Univ. **14** (1974), 1-28

[18] H Sumihiro: *Equivariant Completion II*, J. Math. Kyoto Univ. **15** (1975), 573-605

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